Designs for Correlated Response Models with an Unknown Dispersion Matrix

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Abstract: This paper deals with the comparison of designs for correlated response models with an unknown variance-covariance matrix. The novelty of the proposed approach lies in combining the Quantile Dispersion Graphs (QDGs) technique with the power of the multivariate test for the equality of parameters from such models. The approach is based on considering the quantiles of a certain criterion function (namely, power of three multivariate tests) on concentric surfaces within a particular region of the so-called alternative space. The dependence of these quantiles on the unknown values of the variances and covariances obtained from the variance-covariance matrix, is depicted by plotting the so-called QDGs of the criterion function. These plots provide a clear assessment of the magnitude of the power value associated with a given design. A numerical example is presented for illustration.

Keywords: correlated response models, dispersion matrix, power of multivariate tests, quantile dispersion graphs

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1 Introduction

Various authors discussed the problem of obtaining exact or approximate tests for comparing two independent regression models; see, for example, Ali and Silver (1985) and Conerly and Mansfield (1988). The problem of comparing correlated response models was discussed by Zellner (1962). A major drawback of his procedure is that it requires an estimate of the variance-covariance matrix, $\mathbf{\Sigma}$, to be used instead of $\mathbf{\Sigma}$. To circumvent the problem of an unknown $\mathbf{\Sigma}$, Smith and Choi

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(1982) developed an exact method to compare two correlated response models without having the need to estimate $\mathbf{\Sigma}$. Khuri (1986) introduced a general procedure involving exact multivariate tests for the equality of parameters from several correlated response models with an unknown $\mathbf{\Sigma}$. This procedure assumed the response models to be of the same form and to contain the same set of control variables.

The present article deals with the comparison of designs for correlated response models with an unknown variance-covariance matrix, $\mathbf{\Sigma}$. The novelty of our approach lies in applying the quantile dispersion graphs approach (see Saha and Khuri (2006a) for a review of related literature) in an investigation of the power of several multivariate tests concerning such models (Khuri (1986)). Our proposed approach is based on considering quantiles of a certain criterion function (namely, power of each of three multivariate tests) on concentric surfaces within a particular region of the so-called *alternative space*. The dependence of these quantiles on the unknown values of the variances and correlations obtained from the variance-covariance matrix, $\mathbf{\Sigma}$ is depicted by plotting the so-called *quantile dispersion graphs* (QDGs) of the criterion function. These plots provide a clear assessment of the magnitude of the power value associated with a given design. A numerical example is presented to illustrate the proposed methodology. Power comparisons of the four multivariate tests, namely, Roy's largest root, Wilks' likelihood ratio, Hotelling-Lawley's trace, and Pillai's trace were considered by several authors (see Pillai and Jayachandran (1967), Roy et al. (1971, chap. 5), Seber (1984, sec. 8.6.2)). Pillai (1977) gave an illuminating review of the non-null distributions of characteristic roots in multivariate analysis.

This article is organized as follows: Section 2 introduces correlated response models and the hypothesis of interest. The development of the corresponding multivariate test is discussed in Section 3. The design-criteria functions for comparing designs for such models are developed in Section 4. The QDG approach is explained in Section 5. This is followed by a numerical example in Section 6 to illustrate the proposed methodology. Finally, concluding remarks concerning future research are presented in Section 7.

2 Correlated Response Models

Let us consider a system of r correlated response variables, y_1, y_2, …, y_r, each of which depends on the same p control variables denoted by x_1, x_2, …, x_p. The functional relationship between y_i ($i = 1$, 2, …, r) and x_1, x_2, …, x_p is represented by a linear model of the form

$$y_i = \beta_{i0} + \sum_{k=1}^{p} \beta_{ik} x_k + \epsilon_i, \quad i = 1, 2, \ldots, r, \tag{1}$$

where β_{i0} and the β_{ik}'s are unknown parameters and ϵ_i is a random error associated with the ith response($i = 1$, 2, …, r). The above r models can have different design matrices. Assuming each design consists of n experimental runs and if $\boldsymbol{X}_i$ denotes the $n \times p$ design matrix for the ith response model($i = 1$, 2, …, r), then these r models can be written in vector form as

$$\boldsymbol{y}_i = \beta_{i0}\mathbf{1}_n + \boldsymbol{X}_i\boldsymbol{\beta}_i + \boldsymbol{\epsilon}_i, \quad i = 1, 2, \ldots, r, \tag{2}$$

where $\boldsymbol{y}_i$ is an $n \times 1$ vector of observations on the ith response, $\mathbf{1}_n$ is an $n \times 1$ vector of ones, $\boldsymbol{\beta}_i = (\beta_{i1}, \beta_{i2}, \ldots, \beta_{ip})'$, the matrix $\boldsymbol{X}_i$ is assumed to be of rank p, and $\boldsymbol{\epsilon}_i$ is a vector of random errors associated with the n observations from the ith response($i = 1$, 2, …, r). The following is a multivariate formulation of the above models

$$\boldsymbol{Y} = \mathbf{1}_n\boldsymbol{\beta}_0' + \boldsymbol{X}\boldsymbol{B} + \boldsymbol{\epsilon} \tag{3}$$

where

$$\begin{aligned} \boldsymbol{Y} &= (\boldsymbol{y}_1 : \boldsymbol{y}_2 : \cdots : \boldsymbol{y}_r) \\ \boldsymbol{\beta}_0' &= (\beta_{10}, \beta_{20}, \ldots, \beta_{r0}) \\ \boldsymbol{X} &= (\boldsymbol{X}_1 : \boldsymbol{X}_2 : \cdots : \boldsymbol{X}_r) \\ \boldsymbol{B} &= diag(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_r) \\ \boldsymbol{\epsilon} &= (\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \cdots, \boldsymbol{\epsilon}_r) \end{aligned} \tag{4}$$

are matrices of orders $n \times r$, $1 \times r$, $n \times (pr)$, $(pr) \times r$ and $n \times r$, respectively, and *diag* means that the matrix $\boldsymbol{B}$ is block diagonal. It is assumed that the rows of $\boldsymbol{\epsilon}$ are independent random vectors from multivariate normal distribution $N(\mathbf{0}, \boldsymbol{\Sigma})$ with an unknown variance-covariance matrix $\boldsymbol{\Sigma}$ of order $r \times r$ and rank r.

Let us consider the following hypothesis $H_0 : \beta_{10} = \beta_{20} = \cdots = \beta_{r0}$; $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \cdots = \boldsymbol{\beta}_r$ against $H_a : \beta_{i0} \neq \beta_{l0}$ for some $i \neq l = 1, 2, \ldots, r$; or $\boldsymbol{\beta}_m \neq \boldsymbol{\beta}_u$ for some $m \neq u = 1, 2, \ldots, r$. This is known as the hypothesis of concurrence.

3 Development of a Multivariate Test

Consider the multiresponse model (3). Let ρ denote the rank of $\boldsymbol{X}$. Note that since each $\boldsymbol{X}_i (i = 1,\ 2,\ \ldots,\ r)$ in $\boldsymbol{X}$ is of rank p, we have $p \leq \rho \leq pr$. We shall assume that $r \leq n - \rho$. Let $\boldsymbol{C}$ be the $r \times (r-1)$ matrix of rank $r-1$:

$$\boldsymbol{C} = \begin{bmatrix} 1 & 1 & & 1 \\ -1 & 0 & & 0 \\ 0 & -1 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & -1 \end{bmatrix} \tag{5}$$

From model ((3)) we obtain

$$\boldsymbol{Y}\boldsymbol{C} = \mathbf{1}_n \boldsymbol{\beta}_0' \boldsymbol{C} + \boldsymbol{X}\boldsymbol{B}\boldsymbol{C} + \boldsymbol{\epsilon}\boldsymbol{C} \tag{6}$$

The rows of the matrix $\boldsymbol{\epsilon}\boldsymbol{C}$ in (6) are independent random vectors from the multivariate normal distribution $N(\mathbf{0}, \boldsymbol{\Sigma}_c)$, where $\boldsymbol{\Sigma}_c = \boldsymbol{C}'\boldsymbol{\Sigma}\boldsymbol{C}$. The matrix $\boldsymbol{B}\boldsymbol{C}$ is of order $pr \times (r-1)$ and has the form

$$\boldsymbol{B}\boldsymbol{C} = \begin{bmatrix} \boldsymbol{\beta}_1 & \boldsymbol{\beta}_1 & & \boldsymbol{\beta}_1 \\ -\boldsymbol{\beta}_2 & \mathbf{0} & & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{\beta}_3 & \cdots & 0 \\ \mathbf{0} & \mathbf{0} & & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & & -\boldsymbol{\beta}_r \end{bmatrix} \tag{7}$$

The null hypothesis H_0 can be expressed as

$$H_0 : \boldsymbol{\beta}_0'\boldsymbol{C} = \boldsymbol{0}', \quad \boldsymbol{WBC} = \boldsymbol{0},$$

where $\boldsymbol{W}$ is a matrix of order $p \times (pr)$ of the form

$$\boldsymbol{W} = (\boldsymbol{I}_p : \boldsymbol{I}_p : \cdots : \boldsymbol{I}_p) \tag{8}$$

where $\boldsymbol{I}_p$ is the identity matrix of the order $p \times p$. Model ((6)) and the hypothesis H_0 can be rewritten as

$$\boldsymbol{YC} = \boldsymbol{Z\Gamma} + \boldsymbol{\epsilon C} \tag{9}$$

and

$$H_0 : \boldsymbol{G\Gamma} = \boldsymbol{0} \tag{10}$$

where $\boldsymbol{Z} = (\boldsymbol{1}_n : \boldsymbol{X})$, $\boldsymbol{\Gamma}$ and $\boldsymbol{G}$ are the matrices

$$\boldsymbol{\Gamma} = \begin{bmatrix} \boldsymbol{\beta}_0'\boldsymbol{C} \\ \boldsymbol{bC} \end{bmatrix} \tag{11}$$

and

$$\boldsymbol{G} = \begin{bmatrix} 1 & \boldsymbol{0}_1' \\ \boldsymbol{0}_2 & \boldsymbol{W} \end{bmatrix} \tag{12}$$

where $\boldsymbol{0}_1'$ and $\boldsymbol{0}_2$ are zero vectors of orders $1 \times (pr)$ and $p \times 1$ respectively. Note that $\boldsymbol{G}$ is of full row rank equal to $p + 1$.

The development of a multivariate test for testing H_0 depends on Roy's union-intersection principle. This is illustrated as follows (see Khuri (1986) for details): Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_{r-1})'$ be an arbitrary nonzero vector of order $(r - 1) \times 1$. The multivariate model given in(9) can be reduced to a univariate model by the transformations $\boldsymbol{y}_a = \boldsymbol{YCa}$, $\boldsymbol{\zeta}_a = \boldsymbol{\Gamma a}$, and $\boldsymbol{\epsilon}_a = \epsilon \boldsymbol{Ca}$. Then,

$$\boldsymbol{y}_a = \boldsymbol{Z\zeta}_a + \boldsymbol{\epsilon}_a \tag{13}$$

Note that $\boldsymbol{\epsilon}_a \sim N(\boldsymbol{0}, \sigma_a^2 \boldsymbol{I}_n)$, where $\sigma_a^2 = \boldsymbol{a}'\boldsymbol{C}'\boldsymbol{\Sigma}\boldsymbol{Ca}$. The multivariate hypothesis stated in (10) also reduces to the univariate hypothesis

$$H_0(\boldsymbol{a}) : \boldsymbol{G\zeta}_a = \boldsymbol{0} \tag{14}$$

Clearly, H_0 is true if and only if $H_0(\boldsymbol{a})$ is true for all $\boldsymbol{a} \neq \boldsymbol{0}$. But, for each $\boldsymbol{a} \neq \boldsymbol{0}$, the hypothesis $H_0(\boldsymbol{a})$ is a general linear hypothesis associated with the univariate model (13). The hypothesis can, therefore, be rejected for large values of the statistic

$$R(\boldsymbol{a}) = \frac{\boldsymbol{y}_a'\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^-\boldsymbol{G}'[\boldsymbol{G}(\boldsymbol{Z}'\boldsymbol{Z})^-\boldsymbol{G}']^{-1}\boldsymbol{G}(\boldsymbol{Z}'\boldsymbol{Z})^-\boldsymbol{Z}'\boldsymbol{y}_a}{\boldsymbol{y}_a'\left[\boldsymbol{I}_n - \boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^-\boldsymbol{Z}'\right]\boldsymbol{y}_a} \tag{15}$$

where $(\boldsymbol{Z}'\boldsymbol{Z})^-$ is a generalized inverse of $\boldsymbol{Z}'\boldsymbol{Z}$. This statistic is invariant to the choice of the generalized inverse (see Searle 1971, sec. 5.5). Since $\boldsymbol{y}_a = \boldsymbol{YCa}$, then $R(\boldsymbol{a})$ can be rewritten as

$$R(\boldsymbol{a}) = (\boldsymbol{a}'\boldsymbol{S}_h\boldsymbol{a})/(\boldsymbol{a}'\boldsymbol{S}_e\boldsymbol{a}) \tag{16}$$

where

$$\boldsymbol{S}_h = \boldsymbol{C}'\boldsymbol{Y}'\boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^-\boldsymbol{G}'[\boldsymbol{G}(\boldsymbol{Z}'\boldsymbol{Z})^-\boldsymbol{G}']^{-1}\boldsymbol{G}(\boldsymbol{Z}'\boldsymbol{Z})^-\boldsymbol{Z}'\boldsymbol{YC} \tag{17}$$

and

$$\boldsymbol{S}_e = \boldsymbol{C}'\boldsymbol{Y}'\left[\boldsymbol{I}_n - \boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^-\boldsymbol{Z}'\right]\boldsymbol{YC} \tag{18}$$

The matrices $\boldsymbol{S}_h$ and $\boldsymbol{S}_e$ are called, respectively, the matrix due to the hypothesis and the matrix due to the error. The matrix $\boldsymbol{S}_e$, under the condition $n-\rho-1 \geq r-1$, has the central Wishart distribution $W(n-\rho-1, \boldsymbol{C}'\boldsymbol{\Sigma}\boldsymbol{C})$ with $n-\rho-1$ degrees of freedom. Since $\boldsymbol{G}$ in (10) is of full row rank equal to $p+1$, $\boldsymbol{S}_h$ is independent of $\boldsymbol{S}_e$ and has the noncentral Wishart distribution $W(p+1, \boldsymbol{C}'\boldsymbol{\Sigma}\boldsymbol{C}, \boldsymbol{\Omega})$ with $p+1$ degrees of freedom and noncentrality parameter matrix given by (see Seber 1984, p.414)

$$\boldsymbol{\Omega} = (\boldsymbol{C}'\boldsymbol{\Sigma}\boldsymbol{C})^{-1}\boldsymbol{\Gamma}'\boldsymbol{G}'[\boldsymbol{G}(\boldsymbol{Z}'\boldsymbol{Z})^-\boldsymbol{G}']^{-1}\boldsymbol{G}\boldsymbol{\Gamma} \tag{19}$$

Roy's largest root test statistic is given by (see Khuri (1986, p. 350)) $e_{max}\boldsymbol{S}_h\boldsymbol{S}_e^{-1}$, which denotes the largest eigenvalue of the matrix $\boldsymbol{S}_h\boldsymbol{S}_e^{-1}$. Other multivariate test statistics for testing H_0 include the following:

Wilks's likelihood ratio: $\Lambda = [det(\boldsymbol{S}_e)]/[det(\boldsymbol{S}_e + \boldsymbol{S}_h)]$,

Hotelling-Lawley's trace: $U = tr(\boldsymbol{S}_h\boldsymbol{S}_e^{-1})$, and

Pillai's trace: $V = tr[\boldsymbol{S}_h(\boldsymbol{S}_h + \boldsymbol{S}_e^{-1})^{-1}]$.

3.1 A Special Case

As a special case, let us consider two *independent* regression models,

$$\boldsymbol{y}_i = \boldsymbol{X}_i \boldsymbol{\beta}_i + \boldsymbol{\epsilon}_i, \quad i = 1, 2, \tag{20}$$

where $\boldsymbol{y}_i$ is an $n_i \times 1$ vector of observations on the response variable for the i^{th} model, $\boldsymbol{X}_i$ is a known matrix of order $n_i \times p$ and rank p, and $\boldsymbol{\beta}_i$ is a $p \times 1$ vector of unknown parameters ($i = 1, 2$). It is assumed that the $\boldsymbol{\epsilon}_i$'s are normally distributed random vectors with $E(\boldsymbol{\epsilon}_i) = \mathbf{0}$, $E(\boldsymbol{\epsilon}_i \boldsymbol{\epsilon}_i') = \sigma_i^2 \boldsymbol{I}_{n_i}$ for $i = 1, 2$, and $E(\boldsymbol{\epsilon}_1 \boldsymbol{\epsilon}_2') = \mathbf{0}$. A reasonable hypothesis of interest is $H_0 : \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2$, with the alternative $H_a : \boldsymbol{\beta}_1 \neq \boldsymbol{\beta}_2$. The likelihood ratio test for H_0 when σ_1^2/σ_2^2 is known is based on the test statistic (see for example, Ali and Silver (1985, Eq. 2.2))

$$F = \frac{n - 2p}{p} \times \frac{(\hat{\boldsymbol{\beta}}_1 - \hat{\boldsymbol{\beta}}_2)'[\sigma_1^2(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1} + \sigma_2^2(\boldsymbol{X}_2'\boldsymbol{X}_2)^{-1}]^{-1}(\hat{\boldsymbol{\beta}}_1 - \hat{\boldsymbol{\beta}}_2)}{(n_1 - p)s_1^2/\sigma_1^2 + (n_2 - p)s_2^2/\sigma_2^2} \tag{21}$$

where $n = n_1 + n_2$, $\hat{\boldsymbol{\beta}}_i = (\boldsymbol{X}_i'\boldsymbol{X}_i)^{-1}\boldsymbol{X}_i'\boldsymbol{y}_i$, and $s_i^2 = \frac{1}{n_i - p}(\boldsymbol{y}_i - \boldsymbol{X}_i\hat{\boldsymbol{\beta}}_i)'(\boldsymbol{y}_i - \boldsymbol{X}_i\hat{\boldsymbol{\beta}}_i)$ for $i = 1, 2$. Under H_0, F has an exact F distribution with p and $n - 2p$ degrees of freedom. Under the alternative hypothesis, F is distributed as a noncentral F with p and $n - 2p$ degrees of freedom and noncentrality parameter (see for example, Conerly and Mansfield (1988, p. 815))

$$\tilde{\lambda} = (\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2)'[\sigma_1^2(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1} + \sigma_2^2(\boldsymbol{X}_2'\boldsymbol{X}_2)^{-1}]^{-1}(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2) \tag{22}$$

Note that in this case, knowledge of both σ_1^2 and σ_2^2 is needed for the computation of the power for specified alternative values of $\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2$. This situation is a generalization of the classical Behrens-Fisher problem of testing equality of two means when the population variances are different.

4 Design Criteria

The power of each of the multivariate tests - Roy's largest root, Wilks's likelihood ratio, Hotelling-Lawley's trace, and Pillai's trace (to be described later) will be used as a design criterion. It is well known that the power is a monotone-increasing function of the eigenvalues of the noncentrality matrix, $\mathbf{\Omega}$ (see, for example, Roy et al. (1971, p. 68)). As pointed out by Pillai (1977), the non-null distribution of the Roy's largest root has only been derived in finite or infinite series forms, involving zonal polynomials, whose general terms are difficult to compute. In general, exact power functions of the four multivariate tests are extremely difficult to obtain in closed-form expressions. Since the power of Roy's largest root test is below those of the other three tests (see Pillai and Jayachandran (1967) and Muirhead (1982, sec. 10.6.5)), we consider the power of the other three tests as design criteria. It should be noted that no single test performed better than any of the other tests in terms of power (see Khuri (1986, p. 351) and the references cited therein).

We now present asymptotic formulas that provide adequate approximations for the power of three multivariate tests (see, for example, Khuri (1986)). From (10), the null and the hypothesis can be written as $H_0 : \boldsymbol{G\Gamma} = \mathbf{0}$ and $H_a : \boldsymbol{G\Gamma} = \mathbf{\Delta}$ where $\mathbf{\Delta} = \mathbf{0}$.

4.1 Wilks's likelihood ratio

The power of Wilks's likelihood ratio under the alternative hypothesis H_a and at a level of significance α is approximately given by (see, for example, Khuri (1986, Formula 3.3))

$\Pr(-\delta\ log\Lambda > \tau_W(\alpha)|H_a) = \Pr(\chi^2_f(\sigma_1) > \tau_W(\alpha))$
- $\frac{1}{4\delta}(r+p+1)\sigma_1\ \Pr(\chi^2_{f+2}(\sigma_1) \leq \tau_W(\alpha))$
- $[(r+p+1)\sigma_1 - \sigma_2]\ \Pr(\chi^2_{f+4}(\sigma_1) \leq \tau_W(\alpha))$
- $\sigma_2\ \Pr(\chi^2_{f+6}(\sigma_1) \leq \tau_W(\alpha))$

where $\delta = n - \rho - 1 - (r-p-1)/2$, $\Lambda = [det(\boldsymbol{S}_e)]/[det(\boldsymbol{S}_e + \boldsymbol{S}_h)]$, $\tau_W(\alpha)$ denotes the upper $100\alpha\%$ point of $-\delta\ log\Lambda$ under the null hypothesis H_0, $f = (r-1)(p+1)$, $\sigma_1 = tr(\mathbf{\Omega})$, $\sigma_2 = tr(\mathbf{\Omega}^2)$, $\mathbf{\Omega}$ is the noncentrality parameter matrix defined as above and $\chi^2_f(\sigma_1)$ denotes the noncentral chi-squared variate with f degrees of freedom and noncentrality parameter σ_1.

4.2 Hotelling-Lawley's trace

The power of Hotelling-Lawley's trace under the alternative hypothesis H_a and at a level of significance α is approximately given by (see, for example, Khuri (1986, Formula 3.4))

$$\Pr((n-\rho-1)U > \tau_H(\alpha)|H_a) = \Pr(\chi^2_f(\sigma_1) > \tau_H(\alpha))$$
$$- \tfrac{1}{4(n-\rho-1)}(p+1)(r-1)(p-r+1)\sigma_1 \Pr(\chi^2_f(\sigma_1) \leq \tau_H(\alpha))$$
$$+ 2(p+1)[\sigma_1 - (p+1)(r-1)] \Pr(\chi^2_{f+2}(\sigma_1) \leq \tau_H(\alpha))$$
$$+ [(p+1)(r-1)(p-r+1) - 2(2p+r+2)\sigma_1 + \sigma_2] \Pr(\chi^2_{f+4}(\sigma_1) \leq \tau_H(\alpha))$$
$$+ [2(p+r+1)\sigma_1 - 2\sigma_2] \Pr(\chi^2_{f+6}(\sigma_1) \leq \tau_H(\alpha))$$
$$+ \sigma_2 \Pr(\chi^2_{f+8}(\sigma_1) \leq \tau_H(\alpha))$$

where $U = tr(\boldsymbol{S}_h\boldsymbol{S}_e^{-1})$ is Hotelling-Lawley's trace, $\tau_H(\alpha)$ denotes the upper $100\alpha\%$ point of the statistic $(n-\rho-1)U$, f, σ_1, and σ_2 are the same as defined above.

4.3 Pillai's trace

The power of Pillai's trace under the alternative hypothesis H_a and at a level of significance α is approximately given by (see, for example, Khuri (1986, Formula 3.5))

$$\Pr((n-\rho-1)V > \tau_P(\alpha)|H_a) = \Pr(\chi^2_f(\sigma_1) > \tau_P(\alpha))$$
$$- \tfrac{1}{4(n-\rho-1)}(p+1)(r-1)(p-r+1)\sigma_1 \Pr(\chi^2_f(\sigma_1) \leq \tau_P(\alpha))$$
$$+ [2r(p+1)(r-1) + 2(p+1)\sigma_1] \Pr(\chi^2_{f+2}(\sigma_1) \leq \tau_P(\alpha))$$
$$+ [-(p+1)(r-1)(p-r+1) + 2r\sigma_1 + \sigma_2] \Pr(\chi^2_{f+4}(\sigma_1) \leq \tau_P(\alpha))$$
$$- 2(p+r+1)\sigma_1 \Pr(\chi^2_{f+6}(\sigma_1) \leq \tau_P(\alpha))$$
$$- \sigma_2 \Pr(\chi^2_{f+8}(\sigma_1) \leq \tau_P(\alpha))$$

where $V = tr[\boldsymbol{S}_h(\boldsymbol{S}_h+\boldsymbol{S}_e^{-1})^{-1}]$, $\tau_P(\alpha)$ denotes the upper $100\alpha\%$ point of the statistic

$(n - \rho - 1)V$, f, σ_1, and σ_2 are the same as defined above.

5 Quantile Dispersion Graphs

The performance of a given test is usually measured by its power value under some alternative hypothesis. In this section, we illustrate how the power of a statistical test can be used to compare designs for several correlated response models. The power depends on r (number of responses), the degrees of freedom of $\boldsymbol{S}_h$ and $\boldsymbol{S}_e$, α (level of significance), and the noncentrality parameter matrix $\boldsymbol{\Omega}$ only through its eigenvalues. The construction of designs for these models would therefore require some prior knowledge of $\boldsymbol{\Omega}$. This design dependence problem is also a common feature of designs for variance component estimation, and for designs for nonlinear models, including generalized linear models. There are several approaches to dealing with this design dependence problem, namely, the use of locally optimal designs, the sequential method, the Bayesian methodology, and the use of *quantile dispersion graphs* (QDGs). The last approach is a graphical technique used for comparing designs, and is a key part of our proposed methodology. For an overview of the first three approaches and a detailed discussion regarding the QDG approach, see Khuri (2003), Saha and Khuri (2006a), and the references mentioned therein.

Khuri (1997) first introduced the QDG approach to circumvent the design dependence problem when the design criterion depends on unknown parameters. In this approach, the distribution of a criterion function is determined in terms of its quantiles. Plots of these quantiles depend on the design used and on the true values of the model's unknown parameters. For a given design, the dependence of a particular quantile on the values of the unknown parameters is assessed by computing the maximum and minimum of the quantile over the parameter space of the unknown parameters. Plots of the resulting maxima and minima produce the so-called quantile dispersion graphs (QDGs).

The noncentrality parameter matrix $\boldsymbol{\Omega}$ depends on the design through the matrix

$\boldsymbol{Z} = (\mathbf{1}_n : \boldsymbol{X})$, Σ and $\boldsymbol{G\Gamma} = \boldsymbol{\Delta}$. It should be noted that in order to implement the QDG approach, we need to construct confidence intervals for the elements of $\boldsymbol{\Sigma}$. In addition, the specification of the so-called alternative space for $\boldsymbol{\Delta}$ will be needed. We refer to the example (given later) and computation details for a better understanding of the key steps involved in this approach. Also, refer to Saha and Khuri (2006b) to have a better understanding of the use of parameter space and alternative space.

5.1 Simultaneous Confidence Intervals on the elements of $\boldsymbol{\Sigma}$

In order to obtain simultaneous confidence intervals on the elements of $\boldsymbol{\Sigma}$, we adopt the following procedure given in Seber (1984, Sec. 3.5.7). Let us write $\boldsymbol{\Sigma} = ((\sigma_{jk}))$. By defining $\boldsymbol{Q} = \sum_{i=1}^{r}(\boldsymbol{y}_i - \overline{\boldsymbol{y}})(\boldsymbol{y}_i - \overline{\boldsymbol{y}})'$, where $\overline{\boldsymbol{y}} = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{y}_i$, it can be easily shown that $L \leq \dfrac{\boldsymbol{l}'\boldsymbol{Q}\boldsymbol{l}}{\boldsymbol{l}'\boldsymbol{\Sigma}\boldsymbol{l}} \leq U$, for all nonzero $\boldsymbol{l}$, if and only if $L \leq \gamma_{min} \leq \gamma_{max} \leq U$, where γ_{min} and γ_{max} are the minimum and maximum eigenvalues, respectively, of $\boldsymbol{Q}\boldsymbol{\Sigma}^{-1}$. The readers are referred to Seber (1984, Sec. 3.5.7) in order to have a better understanding about the choice of the values of L and U.

Choosing L and U such that $P[\gamma_{min} \geq L] = \alpha/2$ and $P[\gamma_{max} \leq U] = \alpha/2$, we get $1 - \alpha = P[\frac{\boldsymbol{l}'\boldsymbol{Q}\boldsymbol{l}}{U} \leq \boldsymbol{l}'\boldsymbol{\Sigma}\boldsymbol{l} \leq \frac{\boldsymbol{l}'\boldsymbol{Q}\boldsymbol{l}}{L}]$. Setting $\boldsymbol{l}$ equal to (1,0,...,0), (0,1,...,0), and so on, we obtain the following intervals for the variances and covariances, namely, σ_{11}, σ_{22},....., σ_{rr} and σ_{12}, σ_{13},....., $\sigma_{(r-1),r}$:
$\frac{q_{ii}}{U} \leq \sigma_{ii} \leq \frac{q_{ii}}{L}$ for $i = 1, 2, ..., r$ and
$\frac{q_{ij}}{U} + (q_{ii} + q_{jj})(\frac{1}{U} - \frac{1}{L}) \leq 2\sigma_{ij} \leq \frac{q_{ij}}{L} + (q_{ii} + q_{jj})(\frac{1}{L} - \frac{1}{U})$ for $i < j = 1, 2, ..., r$,
where $q_{ij} = (i, j)^{th}$ element of Q.

5.2 An Alternative Space for $\boldsymbol{\Delta} = \boldsymbol{G\Gamma}$

Let $\boldsymbol{\delta} = \mathbf{vec}(\boldsymbol{\Delta})$, where $\mathbf{vec(.)}$ is a matrix operator (see Searle (1982, pp. 332-333)) which stacks the columns of the matrix, $\boldsymbol{\Delta}$ one under the other to form a single column. This technique was used by Valeroso and Khuri (1999, p. 163). In order to construct an alternative space for $\boldsymbol{\delta} = (\delta_1, ..., \delta_q)'$, which is a subset of R^q, one can select $\boldsymbol{\delta}$ points that fall on a hypersphere of radius τ. Note that here q equals the product of the number of columns and rows in the matrix $\boldsymbol{\Delta}$. For example, if $\boldsymbol{\Delta}$ is a 3×2 matrix, then $q = 3 \times 2 = 6$. Let T_τ denote the surface of such a hypersphere. The points on T_τ depend on only $q-1$ independent variables. Using, for example, the spherical coordinates associated with $\delta_1, ..., \delta_q$, we obtain the following equations (see Khuri et al. (1996), Saha and Khuri (2006b))

$$\begin{aligned}
\delta_1 &= \tau\ cos\ \phi_1 \\
\delta_2 &= \tau\ sin\ \phi_1\ cos\ \phi_2 \\
&\vdots \\
\delta_{q-1} &= \tau\ sin\ \phi_1\ sin\ \phi_2 \cdots\ sin\ \phi_{q-3}\ sin\ \phi_{q-2}\ cos\ \phi_{q-1} \\
\delta_q &= \tau\ sin\ \phi_1\ sin\ \phi_2 \cdots\ sin\ \phi_{q-3}\ sin\ \phi_{q-2}\ sin\ \phi_{q-1}
\end{aligned}$$

where $\phi_1, ..., \phi_{q-1}$ are angles chosen independently such that $0 \le \phi_i \le \pi$ for $i = 1, 2, ..., q-2$, and $0 \le \phi_{q-1} \le 2\pi$. In order to generate points on T_τ, $\phi_1, ..., \phi_{q-1}$ are chosen randomly from independent uniform distributions, namely, $U(0, \pi)$ for $\phi_1, ..., \phi_{q-2}$ and $U(0, 2\pi)$ for ϕ_{q-1}. We can obtain several alternative values for $\boldsymbol{\delta}$ by varying the value of τ.

6 Example

To illustrate our proposed methodology let us consider an example of a repeated measures design given in Kuehl (2000, p. 514). It is based upon an experiment conducted to determine the effects of X_1 = soil compaction and X_2 = soil moisture on the soil microbes activity. Treated soil samples are placed in airtight containers and incubated under conditions conducive to microbial activity. The microbe activity in each soil sample was measured as the percent increase in the production of CO_2 above atmospheric levels. The design used to generate the data was a 3×3 factorial with three levels of soil compaction and three levels of soil moisture. For each treatment, two replicate soil container units were prepared and CO_2 evolution/kg soil/day was recorded on three successive days giving the response values y_1, y_2 and y_3. Table 1 gives the data for each soil container unit and Table 2 shows the coded variables (obtained by subtracting the mean, and then dividing by the standard deviation) of Design 1 and 2, which represent the initial design and a randomly generated design within the region of experimentation, respectively (also, see Fig 1). The QDGs help in comparing these two designs based on the power of the three multivariate tests. The program used to generate the QDGs is written using the R language. High values of the power are obviously desirable. All the three figures, namely Figure 2, Figure 3 and Figure 4 show that the maximum quantiles of D_1 are above those of D_2, for $\tau = 2$. For $\tau = 0.1$ and 1, maximum quantiles of D_1 are either above or same as those of D_2. Hence, D_1 is preferred over D_2.

7 Conclusion and Future Remarks

In a series of articles, Khuri (1992, 1996) considered the analysis of response surface models with random block effects in situations involving a single response. A multiresponse experiment, by definition, is one which involves a number of responses measured for each setting of a group of control variables. In many cases, the subdivision of experimental units into blocks necessitates the addition of block effects to the multirepsonse surface model. The modeling of a multiresponse experiment with random blocks (Valeroso and Khuri (1999)) is a sequel or an extension of Khuri (1992, 1996) in the multiresponse case. Saha and Khuri (2006a, 2006b) compared designs for response surface models with random block effects using the scaled prediction variance and the power of a statistical test as design criteria. Their approach involved the use of QDGs. We intend to extend our proposed methodology to address the problem of comparison of designs for multiresponse models with random block effects.

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Table 1: Design 1 - original variables along with responses

X_1	X_2	y_1	y_2	y_3
1.10	0.10	2.70	0.34	0.11
1.10	0.10	2.90	1.57	1.25
1.10	0.20	5.20	5.04	3.70
1.10	0.20	3.60	3.92	2.69
1.10	0.24	4.00	3.47	3.47
1.10	0.24	4.10	3.47	2.46
1.40	0.10	2.60	1.12	0.90
1.40	0.10	2.20	0.78	0.34
1.40	0.20	4.30	3.36	3.02
1.40	0.20	3.90	2.91	2.35
1.40	0.24	1.90	3.02	2.58
1.40	0.24	3.00	3.81	2.69
1.60	0.10	2.00	0.67	0.22
1.60	0.10	3.00	0.78	0.22
1.60	0.20	3.80	2.80	2.02
1.60	0.20	2.60	3.14	2.46
1.60	0.24	1.30	2.69	2.46
1.60	0.24	0.50	0.34	0.00

Table 2: Design 1 and 2 - Coded Variables

Design 1		Design 2	
x_1	x_2	x_1	x_2
-1.08	-1.14	-0.34	-0.36
-1.08	-1.14	0.91	0.83
-1.08	0.29	0.50	-0.13
-1.08	0.29	-0.32	0.00
-1.08	0.86	0.76	0.64
-1.08	0.86	-0.87	-0.78
0.12	-1.14	0.63	-0.54
0.12	-1.14	0.00	0.74
0.12	0.29	0.43	0.26
0.12	0.29	0.17	-0.67
0.12	0.86	0.10	-0.39
0.12	0.86	0.28	0.78
0.92	-1.14	-0.17	-0.82
0.92	-1.14	0.59	0.55
0.92	0.29	-0.80	-0.13
0.92	0.29	0.13	0.54
0.92	0.86	-0.69	-0.92
0.92	0.86	-0.36	0.45

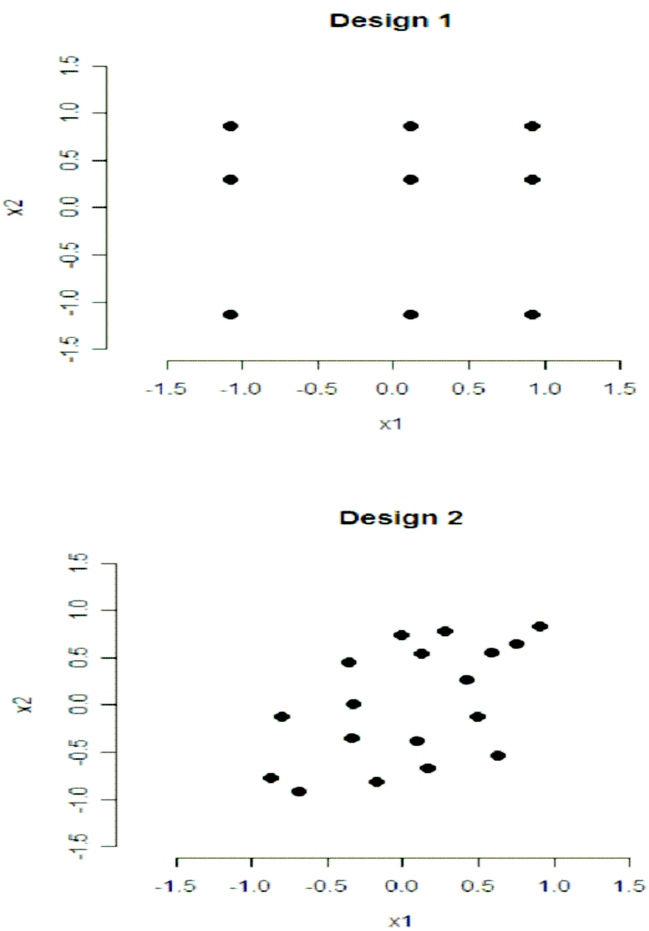

Figure 1: Plot of the design points

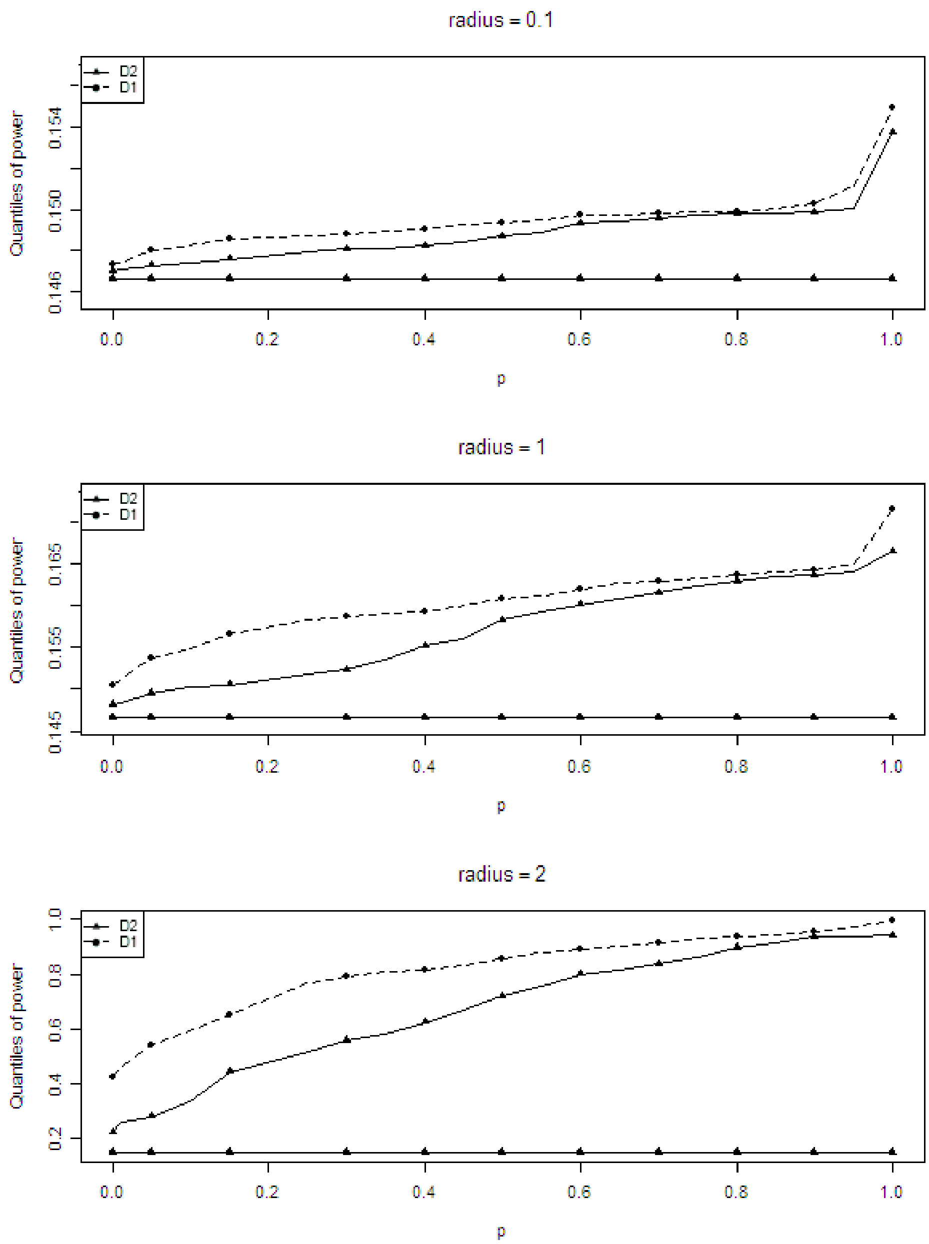

Figure 2: QDGs of the power function for D_1 and D_2 (based on Wilks' Likelihood Ratio)

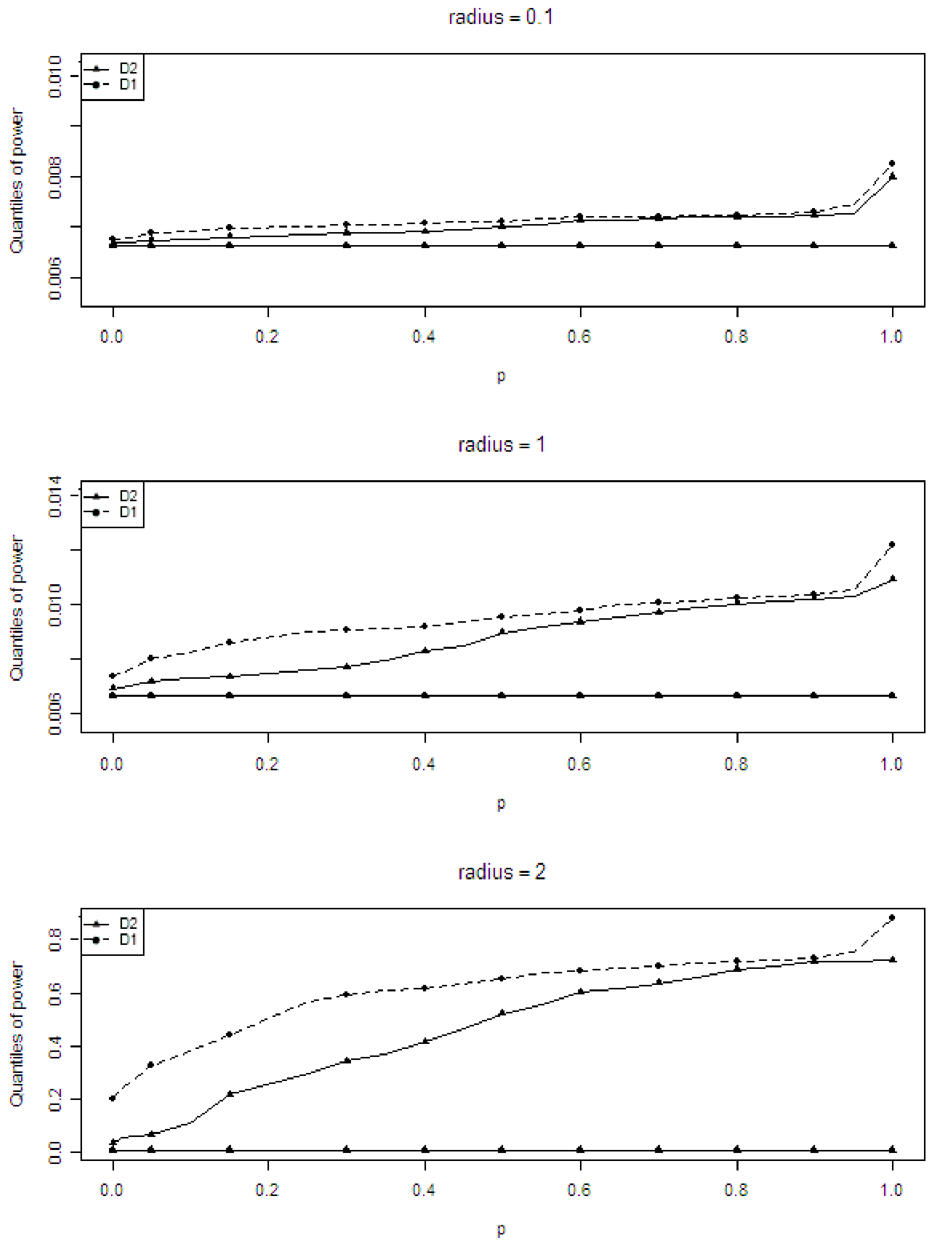

Figure 3: QDGs of the power function for D_1 and D_2 (based on Hotelling-Lawley's Trace)

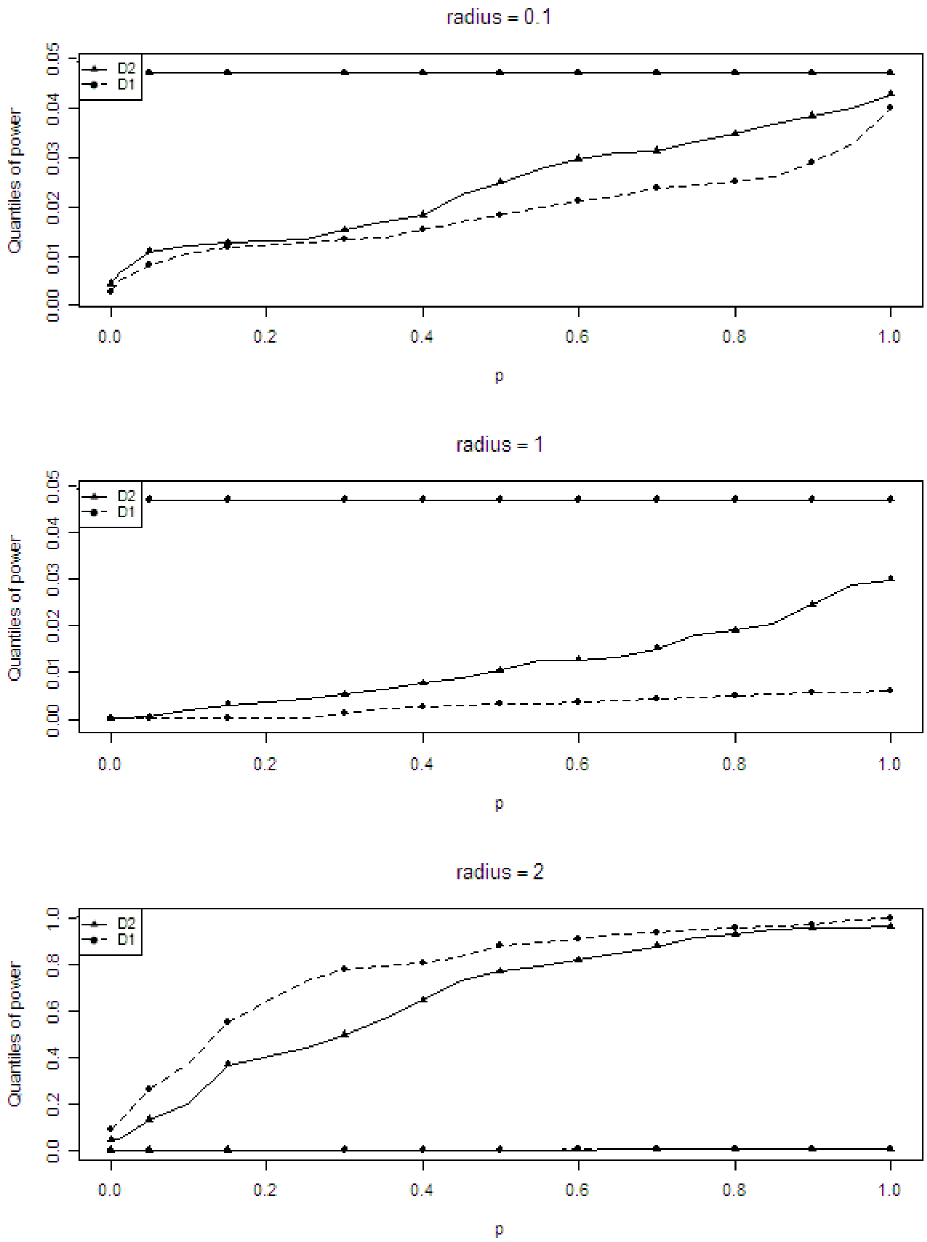

Figure 4: QDGs of the power function for D_1 and D_2 (based on Pillai's Trace)